THE MATHEMATICS TEACHER

EDITED BY
W. H. METZLER

ASSOCIATED WITH

EUGENE R. SMITH — HARRY D. GAYLORD
MAURICE J. BABB — WILLIAM E. BRECKENRIDGE

VOLUME VII — DECEMBER, 1914 — NUMBER 2

GRAMMAR SCHOOL MATHEMATICS.

BY HARRISON E. WEBB.

The opinion is freely expressed nowadays that the present is a transition period in American education. People who hold this view are not agreed as to just what is the nature of the transition. Their judgments on this point are as varied as are their educational creeds. One prominent educator, finding that his favorite psychological hobby will not bear him up the steep and rugged path of mathematics, declares that mathematics ought to be abolished, as far as the schools are concerned. This, to paraphrase Mr. Chesterton, is like trying to get out of having to skin a cat by denying the cat. A world without mathematics is, to say the least, a difficult thing to imagine.

It is likely that the contemporary educational literature of any decade will be found to refer to that period as one of transition. But of the ten years past we may at least say this: that the border lines between college and university mathematics, and between high school and college mathematics have been gradually broken down; and that under the inspiration of many admirable books written to enliven and improve the teaching of mathematics, and of excellent periodicals devoted to the same purpose, and of societies of mathematics teachers which include in their membership representatives of both high schools and colleges, teachers of all grades of mathematical work have come to appreciate the essential unity of their science and the supreme

importance of its mission, from its simplest elements to its most advanced development.

Probably the most important of the results of this improved acquaintance has been, on the one hand, the disposition of high school teachers to extend the bounds of their mathematical knowledge into more advanced fields of study, and on the other, the willingness of college and university instructors to devote much of their time and attention to the consideration of the difficulties which high school teachers have constantly to encounter.

Meanwhile, so far as the writer is able to ascertain, the teacher of mathematics in the grammar school has been left to himself. This is due probably to several conditions.

In the first place the fact is frequently overlooked that there is such a thing as grammar school mathematics. The science itself has been thought to begin with algebra, in the high school. But those whose early mathematical education extends twenty or more years back will have no difficulty in recalling that there was more mathematics, and better mathematics, and harder mathematics in the last of their arithmetic than in the first of their algebra. If this is not true to-day, may it not be due to the fact that certain educational theorists, in the interests of some fantastic lesson-plan, have squeezed most of the mathematics out of arithmetic, leaving what Professor D. E. Smith has well characterized as a mess of historical and economic facts without any vital unity?

In the second place, the conditions under which mathematics is taught in the grammar school make it all but impossible that there should be any vitality in the work. In many, if not most, of American grammar schools, teachers in the seventh and eighth grades are presumed to be equipped to teach mathematics, English, geography, history, elementary science, civics, elocution, and often a half dozen other subjects, and this in spite of the fact that a large majority of these teachers have been denied the privilege of a college education. If it is asked how in the name of great Euclid teachers can be found with sufficient intelligence to give adequate instruction in so many branches, the answer is likely to be given, though not stated quite so baldly, that intelligence is not what is asked of them, as a rule, by those

in authority, but obedience; and that the superior intelligence which inheres in their superiors in administration is able to tell them what to do. The answer *might* be given, however, that in spite of the enormous burden of daily routine, and in spite of the blind adherence of many educational administrators to outworn "principles," many grammar school teachers have by dint of years of hard work so mastered the principles of their mathematics that they have given examples of good teaching which high school and college teachers would do well to emulate.

The content of grammar school mathematics has, however, been left largely to chance, or, what is worse, to psychology. It is fitting, therefore that this field should be thoroughly investigated with reference to the soundness of the fundamental principles which are there presented, and to the appropriateness of their lines of application. It requires, moreover, no lengthy argument to show that the success of high school teaching is inseparably bound up with the right sort of grammar school work in the field of mathematics.

It was with this thought in mind that the writer offered at last winter's meeting of the Middle States Association a resolution looking toward the opening of this question for discussion, his temerity being adequately rebuked by his appointment as chairman of a committee to consider the grammar school situation.

As is usual with committees, its work has progressed slowly owing to more pressing demands upon the time of its membership. But it is safe to anticipate its action in one or two particulars, in the interest of economy of time.

It is desirable that the Association welcome to its membership grammar school teachers and principals who are interested in mathematics; and to the end that the proceedings of the Association be of interest to grammer school teachers, it is to be hoped that questions which relate particularly to their field of work be brought before the Association, and that the grammar school point of view be presented as forcefully as may be. It would be as impractical and as discourteous as well, for high school teachers to attempt to dictate to the grammar schools in this business, as it seemed not so very long ago when the colleges felt free to dictate to the high schools the character of their mathematical work.

It is believed, moreover, that recent investigations into the fundamental principles of mathematics, and the bringing to American teachers of an idea of remarkable work in this field which has for some time past been in progress in the European countries may be found to have an important bearing upon elementary method and content, in the field of grammar school work, as it has already been found to have in the case of high school mathematics.

The problem of the grammar school teacher is to a great extent one of administration. Into this we cannot enter in detail, but one may safely urge that wherever possible, something be done to enable the grammar school teacher to narrow the field of his instruction.

In line with this desired change, a plan has recently been urged by men of experience in school administration which involves the reorganization of the school systems of the country on the basis of a new division of the school period into six years of grammar school and six years of high school. This plan has met with widespread approval as a theory, and has been pursued successfully in some localities. The various departmental associations, such as our own, are not likely to be called upon to pass upon this plan. But we should be wide-awake as to its possibilities, and have in mind some definite recommendations as to the character of a six year course.

This plan leaves the earlier years of mathematical teaching about as they are now, presumably. They in time will demand consideration also. But as the seventh and eighth years are at present the center of interest, the committee has addressed itself to the task of ascertaining what is the present content of the mathematics of those years, and at the same time, what work would be suitable for those years under the "six-and-six" plan.

The question as to whether algebra should be taught at all in the seventh or eighth years is apparently still an open one. In some schools a certain amount of algebra has been made a part of the course of study, only to be discarded later. It is the opinion of the committee that this question should not be decided on general principles, without attention to detail, but that the whole situation should be subjected to a searching analysis.

Among the questions which arise is the one as to what con-

stitutes algebra in this connection. It is to be noted that the arithmetical operations upon letters may be taught without reference to the solution of equations; as for example, in the formulation of rules and principles. Also that negative number may be considered apart from literal forms and from equations. Even the axioms may be applied to the solution of literal problems in equation form without any considerable knowledge of solution of equations in the abstract. In the field of arithmetic proper, an examination of the problems shows a surprisingly large number of topics which (not all in one school, usually) are capable of inclusion. It appears also that there are many diverse points of emphasis to be considered. A glance at the whole gives an impression that proper instruction in arithmetic alone in the grammar school constitutes in itself a very difficult problem.

The writer is strongly of the opinion that familiarity with literal quantities, with the laws of exponents, and with the operations upon negative number, as well as the solution of simple problems by means of equations may well be included in the seventh and eighth years of the grammar school, together with the inversion of simple formulae by algebraic methods; while the ninth or first high school year of mathematics should be devoted in part at least to certain portions of the arithmetic.

He also feels that there is need for much greater emphasis upon the *principles* of arithmetic and algebra, and their definite formulation.

The question of grammar school geometry is also of interest. A striking example of the loose thinking of eminent educators who are not teachers of mathematics is to be found in the oft-repeated statement that geometry should precede algebra in point of time, because it is "more concrete." Here again there is a question of what is meant by geometry. If geometry is taken to mean what it never meant to its founders, viz., measurement of lines in more directions than one, with measurement of areas and volumes, that sort of "geometry" may be easier than some parts of algebra. But if by geometry is meant the study of the principles of bidimensional and tridimensional magnitudes in their essential relations, it is hard to see wherein it can possibly be simpler than an elementary science which is limited to linear

magnitude. As a matter of fact, the testimony of hundreds of students is to the effect that "algebra was easy, but this geometry is terrible." Even from the point of view of interest, wherein, it might be asked, is a triangle more interesting than a radical sign?

Finally, and as of the greatest importance, arises the question as to the way in which high school teaching should relate itself to what has been given previously in the grammar school. It would appear that a careful study of the grammar school situation is likely to influence high school teaching very profoundly.

CENTRAL HIGH SCHOOL,
NEWARK, N. J.

[The committee has prepared a questionnaire, to be accompanied by the following letter, which sufficiently explains its purpose.]

It has now come to be widely recognized that the transfer of a pupil from one school to another of more advanced grade and of somewhat different aims and purposes usually entails some loss of his time and waste of his energy.

Convenience, however, has decreed that the school training of a boy or girl between the (average) ages of six and eighteen shall be divided into two parts. It is important that the work of these two parts should be related as closely as possible. Neither can well be regarded as more important than the other; nor is either in a position to prescribe the limits which shall be set for the other.

In America it is an all but universal custom that the first eight years of the training of a boy or girl shall be spent in primary and grammar school, and the last four years in high school. It is now proposed in many quarters that the change from one type of school to the other shall be made two years earlier (the six-and-six plan).

It is fitting that the Association of Mathematics Teachers of the Middle States and Maryland should devote some time and attention (*a*) to the mathematical work of the seventh and eighth years generally, and (*b*) to possible mathematical courses under the proposed plan. For the purpose of laying these matters before the association a committee has been appointed by the president of the association.

In order to collect data on which to proceed the committee has prepared a questionnaire. It is proposed to submit the questions to

(*a*) High school principals and teachers, to ascertain their opinions as to what work should precede the usual high school course in algebra.

(*b*) Grammar school principals and teachers, to ascertain the proper scope of the work of these two years, in their opinion.

(*c*) Administrators, who have encountered this problem and acted accordingly.

(*d*) Heads of normal schools and colleges, who are responsible for the preparation of teachers.

It is desired that each person should submit two sets of answers, the first relating to the present "eight-and-four" plan, and the second to the proposed "six-and-six" plan.

It is the firm conviction of the committee that the success of either plan is largely in the hands of the teachers, and that general exchange of opinion is highly important at this time.

Your co-operation is earnestly requested, in the form of an early reply.

Yours very truly,

GEORGE B. GERMANN,
Principal, Public School 130, Brooklyn, N. Y.,
HOWARD F. HART,
Montclair High School, Montclair, N. J.,
AGNES LONG,
William Penn High School, Philadelphia, Pa.,
JOHN H. MINNICK,
Horace Mann School, New York, N. Y.,
C. P. SCOBORIA,
Polytechnic Prep. School, Brooklyn, N. Y.,
H. E. WEBB, *Chairman*,
Central High School, Newark, N. J.

QUESTIONNAIRE.

(*a*) The questions given below are intended to refer specifically to the teaching of mathematics in the seventh and eighth school years, but not to preclude consideration of the same topics in years below the seventh.

(*b*) The term "mathematics" is extended to include practical applications to commercial and scientific problems of common experience, without detailed study of the underlying theory of such problems.

(*c*) For convenience the topics and principles which are usually included under "beginning algebra" are divided into two groups; the first called *literal forms*, and including applications of arithmetical principles to literal numbers in combination; and the second called *equations*, and including the solution of problems by the use of one or more unknown quantities. Arithmetical problems solved by equations should be included in this last named group.

(*d*) It is desired that at least two copies of each question paper be returned, the first to relate to the seventh and eighth grades of grammar school, and the second to the first two years in high school under the "six-and-six" plan. Other copies will be furnished and welcomed in return, if it is desired to differentiate between courses with various objects in view.

(*e*) Questions may be answered briefly by the words "Yes" or "No"; or, in the case of topics, by writing the characters 7, 8, 7–8, or × in the proper spaces to signify that topics should be taken in the seventh grade, the eighth grade, both grades, or not at all; but more extended expression of opinion will be welcomed by the committee.

1. Should the entire matter of grammar school arithmetic be covered either in review or as new work (*a*) in the seventh grade?.....(*b*) in the eighth grade?.....(*c*) in the two grades together?.....

2. On a basis of ten points as a maximum, zero as a minimum, indicate the relative importance of the following aspects of arithmetical teaching in the seventh and eighth years:

	7th.	8th.	
(1) Clear comprehension of essential principles	.....	.....	....
(2) Ability to apply principles to unfamiliar cases	.....	.....	....

(3) Accuracy in calculation ....
(4) Rapidity in calculation ....
(5) Mastery of short methods in calculation . ...
(6) Neatness in appearance of work ....
(7) Ability to perform operations mentally .. ..
(8) Literal memory of rules ....
(9) Analytic insight into problems ....
(10) Ability to write solutions in symbolically.. ....
(11) Ability to state solutions in general terms . ...
(12) Practical measurement ....
(13) Estimating of results ....
(14) Checking of results to insure accuracy.... ....
(Please add any others.)

3. Should any rules be taught *verbatim?* If so, which of the following? Rules for

	7th.	8th.	
(1) Addition and subtraction of decimals.....	.....	.....	....
(2) Multiplication of decimals	.....	.....	....
(3) Division of decimals	.....	.....	....
(4) Reduction of common fractions	.....	.....	....
(5) Addition and subtraction of common fractions	.....	.....	....
(6) Multiplication of common fractions	.....	.....	....
(7) Division of common fractions	.....	.....	....
(8) Reduction of fraction to mixed number ..	.....	.....	....
(9) Reduction of mixed number to a fraction	.....	.....	....
(10) Tests for exact divisors..2..3..4..5..6..8 ..9..10	.....	.....	....
(11) General principles of division (any)	.....	.....	....
(12) Highest common factor	.....	.....	....
(13) Lowest common multiple	.....	.....	....
(14) Euclidean method of H. C. F. and L. C. M.	.....	.....	....
(15) Reduction of denominate numbers.....	.....	.....	....
(16) Operations on denominate numbers	.....	.....	....
(17) Longitude and time	.....	.....	....
(18) Square root	.....	.....	....
(19) Percentage generally	.....	.....	....
(20) Law of the right triangle	.....	.....	....
(21) Mensuration rules generally	.....	.....	....

(Please add any others.)

4. Should literal notation be taught (*a*) in the seventh grade? .....(*b*) in the eighth grade?.....

5. Should algebraic solution of equations be taught (*a*) in the seventh grade?.....(*b*) in the eighth grade?.....

6. Should literal problems (not equations) be presented (*a*) in connection with numerical problems of the same class? .....(*b*) separately, as a review of arithmetical principles?.....

7. Should problems in the evaluation of formulæ be given when the formulæ themselves cannot be derived by the students?.....

8. Should the idea of negative numbers be presented (*a*) in the seventh grade?.....(*b*) in the eighth grade?.....

9. Should the idea of approximate values of magnitudes and percentage of error be clearly defined (*a*) in the seventh grade? .....(*b*) in the eighth grade?.....

10. (Please answer the following as suggested in note (*e*) above.) Should the following topics be presented (assuming that the algebraic examples are to be extremely simple)? If so in which year?

	Numerically.	Literally.	In Equations.
Notation and numeration ...	.....	.....	.....
Addition of integers	.....	.....	.....
Subtraction do	.....	.....	.....
Multiplication do	.....	.....	.....
Division do	.....	.....	.....
Decimals	.....	.....	.....
U. S. Money	.....	.....	.....
Reduction of fractions	.....	.....	.....
Operation on fractions	.....	.....	.....
Factoring	.....	.....	.....
H. C. F.	.....	.....	.....
L. C. M.	.....	.....	.....
Average	.....	.....	.....
Ratio and simple proportion .	.....	.....	.....
Compound proportion	.....	.....	.....
Partitive proportion	.....	.....	.....
Partnership	.....	.....	.....
Alligation	.....	.....	.....
Specific gravity	.....	.....	.....
Long measure	.....	.....	.....
Square measure	.....	.....	.....
Cubic measure and capacity .	.....	.....	.....
Dry measure	.....	.....	.....
Liquid measure	.....	.....	.....
Avoirdupois weight	.....	.....	.....

Troy and apothecary's wt.[1]
Metric system
English money
French money
German money
Circular measure
Longitude and time
Board measure
Plastering and painting
Temperature scales
Percentage
Profit and loss
Commission
Commercial discount
Insurance
Life insurance
Taxes
Stocks and bonds
Duties and customs
Simple interest
Exact interest
Bank discount
True discount
Domestic exchange
Foreign exchange
Equation of payments
Powers and exponents
Square root
Cube root
Mensuration of plane figures
  Rectangles
  Parallelograms
  Triangles
  Trapezoids
  Circles
  Regular polygons
Mensuration of solid figures.
  Rectangular solids
  Prisms
  Pyramids
  Cylinders
  Cones
  Spheres
Levers

[1] *Note.*—Many of the topics mentioned are palpably unfit for algebraic treatment. But an examination of the exercises in almost any text-book designed for first year high school work shows the inclusion of a surprisingly large number of topics.

Pulleys ....
Wheel and axle ....
Inclined plane ....
Similar plane figures ....
Similar solids ....
Uniform motion ....
Accelerated motion ....

11. Should the arrangement of the course be mainly topical (*a*) in the seventh grade?.....(*b*) in the eighth grade? .....

12. Should the plotting of graphs of statistics be taught (*a*) in the seventh grade?.....(*b*) in the eighth grade? .....

13. Should the plotting of graphs of simple algebraic functions be taught (*a*) in the seventh grade?.....(*b*) in the eighth grade?.....

14. Should simple exercises in concrete geometry with geometrical drawing be taught (*a*) in the seventh grade?..... (*b*) in the eighth grade?.....

15. Should geometrical drawing be correlated (*a*) with drawing?.....(*b*) with mathematics?.....

16. Should mathematical exercises in the seventh and eighth grades be standardized? (*a*) according to relative difficutly of underlying concepts?.....(*b*) according to the number of operations entering the particular exercises?.....

Please add any further suggestions.

GEOMETRY OF FOUR DIMENSIONS.

By Henry P. Manning.

I have selected a few theorems and will give the proofs of some of them in order that you may know how we study this geometry and just how simple it is.*

I will suppose that we have started with *points* and regard all figures as consisting of points. I assume a relation among points which may be called the *collinear relation,* and define *lines* as well as *planes* and *hyperplanes* by means of this relation. The line is determined by two points, the plane by three points not points of one line, and the hyperplane by four points not points of one plane. The hyperplane, for example, consists of the points that we get if we take four points not points of one plane, all points collinear with any two of them, and all points collinear with any two obtained by this process. Our space is a hyperplane and geometry in a hyperplane is the ordinary solid geometry.

Now in order to get a *space of four dimensions* it is only necessary to assume the existence of five points not in one hyperplane. Space of four dimensions consists of the points that we get if we take five points not in one hyperplane, all points collinear with any two of them, and all points collinear with any two obtained by this process. For convenience I will assume that all points lie in one space of four dimensions and call this space ***hyperspace.***

Theorem 1.—*If two points of a line lie in a hyperplane the line lies entirely in the hyperplane, and if three non-collinear points of a plane lie in a hyperplane the plane lies entirely in the hyperplane.*

For the process of obtaining the line or plane is but a part of the process of obtaining the hyperplane.

* These theorems are given with full details in the "Geometry of Four Dimensions" soon to be published by The Macmillan Company.

THEOREM 2.—*One and only one hyperplane contains*

(1) *a plane and a point not in it, or an intersecting plane and line;*

(2) *two lines not in one plane;*

(3) *three lines through a point not in one plane;*

(4) *two planes intersecting in a line.*

From (4) it follows that two planes through a point but not in one hyperplane can have only one common point.

We might stand at this point and look through a tube and follow one plane completely around without seeing anything of the other plane, and then, turning in some direction on the second plane follow it completely around without seeing anything of the first. A plane extends in two directions, but there are two directions in which it does not extend. We can go completely around a plane just as in ordinary space we can go around a line. However, if we insist that we must see the figures of our geometry we shall stop right here. We can go on only with our reason. The theorem is incontestable, but the picture impossible.

On the other hand, the next three theorems will be readily accepted, although they are much more difficult to prove.

THEOREM 3.—*Two planes in a hyperplane with a point in common have a line in common.*

THEOREM 4.—*A plane and hyperplane with a point in common have a line in common.*

THEOREM 5.—*Two hyperplanes with a point in common have a plane in common.*

I will proceed to take up various kinds of perpendicularity.

A *line is perpendicular to a hyperplane* in much the same way that a line is perpendicular to a plane in solid geometry. We have three theorems:

THEOREM 6.—*The lines perpendicular to a given line m at a point O do not all lie in one plane.*

Proof.—Every point of hyperspace lies in a plane with m, and every plane containing m contains a perpendicular to m at O. If the perpendiculars were all in a plane α, every plane through m would contain two lines of the hyperplane determined by m and α, and so every point of hyperspace would be in this hyperplane.

THEOREM 7.—*A line m perpendicular at a point O to three lines not in one plane is perpendicular to every line through O in the hyperplane of the three lines.*

Proof.—Let *a*, *b*, and *c* be the three lines and let *d* be any other line through *O* in the hyperplane of these three. The plane *cd* intersects the plane *ab* in a line *h* (Th. 3). *m* is perpendicular to the plane *ab* and so to *h*, and then to the plane *hc* and so to *d*.

THEOREM 8.—*All the perpendiculars to a line at a point lie in a single hyperplane.*

Proof.—Let *m* be perpendicular to all the lines through *O* in a given hyperplane as in the preceding proof, and let *d* be given now as perpendicular to *m* at *O*. The plane *md* will intersect the hyperplane in a line *d'* (Th. 4), and in this plane we have *d* and *d'* both perpendicular to *m* at *O*. *d* must therefore coincide with *d'* and lie in the hyperplane.

A line and hyperplane are perpendicular at a point *O* when the line is perpendicular to all the lines of the hyperplane which pass through *O*.

THEOREM 9.—*A line perpendicular to a hyperplane at a point is perpendicular to all the planes of the hyperplane which pass through this point, and all the planes perpendicular to a line at a point lie in a perpendicular hyperplane.*

THEOREM 10.—*Through any point passes one and only one line perpendicular to a given hyperplane, or hyperplane perpendicular to a given line.*

THEOREM 11.—*Two lines perpendicular to a given hyperplane lie in a plane and are parallel.*

For they lie in a hyperplane (see Th. 2 (2)) and are both perpendicular to the plane in which their hyperplane intersects the given hyperplane.

In regard to planes we have three theorems somewhat like the three given to introduce perpendicular lines and hyperplanes (Ths. 6–8):

THEOREM 12.—*A plane has more than one line perpendicular to it at a given point.*

For the plane is the intersection of hyperplanes in each of which there is such a line.

THEOREM 13.—*Two lines perpendicular to a plane at a point O determine a second plane such that every line through O in either of these planes is perpendicular to every line through O in the other.*

THEOREM 14.—*All the perpendiculars to a plane at a given point lie in a single plane.*

Proof.—Let α and α' be two planes intersecting at O, every line of α through O perpendicular to every line of α' through O, and let d be a line through O perpendicular to α. α and d lie in a hyperplane (Th. 2 (1)) which intersects α' in a line d' (Th. 4), and in this hyperplane we have d and d' both perpendicular to α at O. d must therefore coincide with d' and lie in α'.

Two planes intersecting at a point O are *absolutely perpendicular* when every line of one through O is perpendicular to every line of the other through O.

THEOREM 15.—*Through any point passes one and only one plane absolutely perpendicular to a given plane.*

THEOREM 16.—*Two planes absolutely perpendicular to a third lie in a hyperplane and are parallel.*

Proof.—Let α and β be two planes absolutely perpendicular to a plane γ at points O and O', let c be the line OO', and let a and b be the lines through O and O' perpendicular to the hyperplane determined by α and c.* a lies in γ because a is perpendicular to α (Th. 14). b lies in γ because a and b lie in a plane (Th. 11) and γ is the only plane that contains a and O'. b is, then, perpendicular to β and β lies in the hyperplane determined by α and c (Th. 9). In this hyperplane we have α and β perpendicular to c and so parallel.

THEOREM 17.—*If two planes intersect in a line their absolutely perpendicular planes at any point O of their intersection intersect in a line.*

For they are both perpendicular to the line of intersection of the two given planes and so lie in the hyperplane perpendicular to this line at O (Ths. 9 and 3).

Two planes are perpendicular when they lie in a hyperplane and form right dihedral angles.

THEOREM 18.—*A plane perpendicular to one of two absolutely perpendicular planes at their point of intersection is perpendicular to the other.*

Proof.—Let α and α' be two planes absolutely perpendicular

* It is not to be supposed that the figure shows how two planes absolutely perpendicular to a third look.

at a point O, and let β be perpendicular to α along a line which goes through O. Then β contains a line perpendicular to α at O, and so intersects α' in a line and lies in a hyperplane with α' (Ths. 14 and 2 (4)). Now the intersection of β and α, like all the lines of α through O, is perpendicular to α'. Therefore β, lying in a hyperplane with α', and containing a line perpendicular to α', is itself perpendicular to α'.

THEOREM 19.—*A plane intersecting two absolutely perpendicular planes in lines is perpendicular to both.*

THEOREM 20.—*Two planes intersecting in a line have at any point O of their intersection two common perpendicular planes, planes absolutely perpendicular to each other.*

Proof.—The two planes lie in a hyperplane (Th. 2 (4)) and their absolutely perpendicular planes at O intersect in a line and lie in a hyperplane. A common perpendicular plane is a plane which intersects all four of these planes in lines. Now a plane intersecting the two given planes in lines must lie in their hyperplane (Th. 1) or contain their line of intersection, and a plane intersecting in lines the planes absolutely perpendicular to the given planes must lie in their hyperplane or contain their line of intersection. A plane cannot lie in a hyperplane with two of these planes and contain the line of intersection of the other two, for the line is perpendicular to the hyperplane (Th. 7). Such a plane must, therefore, lie in both hyperplanes and be their plane of intersection, or contain both lines of intersection. The common perpendicular plane with which we are familiar, the plane in the hyperplane of the two given planes perpendicular to their intersection, is the plane of intersection of their hyperplane and the hyperplane of their absolutely perpendicular planes at O. The plane determined by the two lines of intersection is absolutely perpendicular to this plane (Th. 13), and these two are the only common perpendicular planes that the two given planes can have.

A *rectangular system* consists of four lines, six planes, and four hyperplanes. The lines are mutually perpendicular; the planes are three pairs of absolutely perpendicular planes, the two planes of any pair perpendicular to each of the four planes of the other two pairs. This system is the basis of rectangular coordinates.

A *plane and hyperplane are perpendicular* at a point O when the absolutely perpendicular plane lies in the hyperplane.

THEOREM 21.—*If a plane and hyperplane, intersecting in a line, are perpendicular at one point of their intersection, they are perpendicular all along this line.*

See proof of Th. 16.

We have several theorems like those in solid geometry about perpendicular planes; for example,

THEOREM 22.—*If a plane and hyperplane are perpendicular, any line in one perpendicular to their intersection is perpendicular to the other.*

THEOREM 23.—*If a plane and hyperplane are perpendicular, a plane in the hyperplane containing the intersection is perpendicular to the plane, and a plane in the hyperplane perpendicular to the intersection is absolutely perpendicular to the plane.*

Hyperplane angles are like ordinary dihedral angles. A *half-hyperplane* is that part of a hyperplane which lies on one side of any one of its planes, and the plane is the *face* of the half-hyperplane. A *hyperplane angle* is formed by two half-hyperplanes with a common face. A *plane angle* of a hyperplane angle is an angle with its sides in the two half-hyperplanes perpendicular to the face. The plane angle measures the hyperplane angle in the same way as with dihedral angles.

THEOREM 24.—*A hyperplane perpendicular to the face of a hyperplane angle intersects it in a dihedral angle with the same plane angle.*

A *plano-polyhedral angle* is formed by half-planes having a common edge and passing through the points of a plane polygon which does not lie in a hyperplane with the edge. The half-planes which pass through the vertices of the polygon are the *faces* of the plano-polyhedral angle. Two successive faces form a dihedral angle and the half-planes which pass through the points of a side of the polygon fill the interior of one of these dihedral angles. There are also hyperplane angles corresponding to the angles of the polygon.

THEOREM 25.—*A hyperplane intersecting the edge but not containing it intersects the plano-polyhedral angle in a polyhedral angle whose face angles and dihedral angles correspond respectively to the dihedral angles and hyperplane angles of of the plano-polyhedral angles.*

When the hyperplane is perpendicular to the edge the corresponding angles have the same measures. Thus we have many theorems analogous to the theorems of polyhedral angles. For example,

THEOREM 26.—*The sum of two dihedral angles of a plano-trihedral angle is greater than the third.*

We may think of a plano-polyhedral angle as generated by a polyhedral angle moving off in a direction away from its hyperplane.

A *polyhedroidal angle* is formed with a vertex and directing-polyhedron. It has face angles, dihedral angles, polyhedral angles, etc., corresponding to the edges, face angles, faces, etc., of the directing-polyhedron.

I come next to the angles of two planes. The planes and lines considered in this connection all pass through a point O. This will be left understood and not mentioned each time.

THEOREM 27.—*Given two planes, α and β, there is a half-line m of α whose angle with β is less than or equal to the angle made with β by any other half-line of α.*

The proof of this theorem involves the principle of continuity.

THEOREM 28.—*The plane of the minimum angle which a half-line of α makes with β is perpendicular also to α.*

Proof.—Let n be the projection of m upon β. Then if m were not the projection of n upon α, the line which is its projection would make with n, and so certainly with β, a smaller angle.*

The plane absolutely perpendicular to mn is another plane perpendicular to α and β (Th. 18). Two planes having a point in common always have two common perpendicular planes. The acute angles formed in these two planes are *the angles of the planes α and β.*

THEOREM 29.—*When the angles of two planes are equal the planes have an infinite number of common perpendicular planes on all of which they cut out the same angles, and any two of the common perpendicular planes cut out angles on β equal to the angles which they cut out on α.*

* This proof is given by C. J. Keyser, "Concerning the Angles and the Angular Determination of Planes in 4-space," *Bulletin of the American Mathematical Society*, Vol. 8, 1902, pp. 324–329.

When two planes have equal angles they are *isocline*. Their common perpendicular planes are also isocline. We have two *series of isocline planes, conjugate series,* all the planes of one series perpendicular to all the planes of the other series. There are two senses in which planes can be isocline; the planes of one of two conjugate series are isocline *positively* and the planes of the other series *negatively*. Isocline planes have many of the properties of parallel lines. For example,

THEOREM 30.—*If a plane (through O) intersects two isocline planes in lines the alternate interior dihedral angles are equal.**

I will mention some of the *hypersolids,* giving only a brief informal statement of a few of the theorems. The solids which go to form a hypersolid are *cells.*

A *hyperpyramid* has a polyhedron base. It may be cut from a polyhedroidal angle.

When the base is a tetrahedron the hyperpyramid is a pentahedroid and can be regarded as a hyperpyramid in five different ways. The five tetrahedrons can be cut apart sufficiently to be spread out in a hyperplane where we can see them all. Four of them rest upon the four faces of the fifth and can be folded on these faces and brought together again so as to enclose a portion of hyperspace.

A hyperpyramid with pyramid base can be regarded in two ways as such a hyperpyramid. It can also be regarded as having a vertex-edge and polygon base, and as cut from a plano-polyhedral angle by two hyperplanes. We will call it a *double pyramid.*

Similarly, we have *hypercones* and *double cones* cut from *hyperconical* and *plano-conical hypersurfaces,* a double cone being also in two ways a hypercone with a cone for base.

A *prismoidal hypersurface* is formed by a system of parallel lines (*elements*) with a directing-polyhedron. A *hyperprism* is cut from such a hypersurface by two parallel hyperplanes.

A *plano-prismatic hypersurface* is formed by a system of parallel planes (*plane elements*) with a directing-polygon, each element intersecting the plane of the polygon only in a single

* These properties of planes are developed by Stringham with the aid of quarternions, "On the Geometry of Planes in a Parabolic Space of Four Dimensions," *Transactions of the American Mathematical Society,* Vol. 2, 1901, pp. 183–214.

point. The elements which contain the vertices of the polygon are the *faces,* and the elements which lie between two successive faces fill up a *cell.* Any plane intersecting the elements only in single points intersects the hypersurface in a polygon which may be taken as directing-polygon.

When the elements of a plano-prismatic hypersurface intersect the elements of a second plano-prismatic hypersurface only in points the intersection of the two hypersurfaces is made up in two ways of the lateral surfaces of a set of prisms. There are two sets of prisms, the prisms of one set being joined crosswise to the prisms of the other set. These two sets of prisms together with their interiors enclose a portion of hyperspace and form a figure which may be called a *double prism.* In a right double prism where the elements of one hypersurface are absolutely perpendicular to the elements of the other, the prisms of either set can be cut apart from those of the other set and spread out in a hyperplane so as to form a single prism with altitude equal to the perimeter of any base of the prisms of the other set. The directing-polygons of either hypersurface which lie in the elements of the other are the *directing-polygons of the double prism.* Any two polygons intersecting in a single point and lying in planes which have only this point in common are directing-polygons of a double prism. The double prism may be generated by moving each of these polygons with its interior around the other, keeping it always parallel to itself.

A hyperprism whose bases are prisms can be regarded in two ways as a hyperprism of this kind. It can also be regarded as a double prism, one set of prisms being parallelopipeds and the other set a set of four.

Similarly, we have *hypercylindrical* and *plano-cylindrical hypersurfaces,* and we have *hypercylinders, prism cylinders* and *double cylinders.* A prism cylinder is formed from a plano-prismatic and a plano-cylindrical hypersurface, and a double cylinder from two plano-cylindrical hypersurfaces. A polygon and a circle intersecting in a point and lying in planes which have only this point in common can be used to generate a prism cylinder, and two circles in the same way generate a double cylinder. Two circles in planes which are absolutely perpendicular determine a double cylinder of double revolution. The

surface of intersection of the two plano-cylindrical hypersurfaces in this case is a surface of some importance in the theory of complex variables.

A *hypersphere* is cut by a hyperplane in a sphere. There are *great spheres* and *small spheres*. Any sphere in the hypersphere has two *poles* and any circle has a *polar great circle*. A great circle is itself polar circle to its polar.

BROWN UNIVERSITY,
PROVIDENCE, R. I.

BLANK FORMS IN ALGEBRA.*

By Albert Harry Wheeler.

Originals in geometry are, perhaps, the best means to be had by the teacher for determining the grasp the pupils have on the subject and for teaching geometry in the true sense.

Originals in algebra are not commonly recognized as a class of examples by themselves, and the great bulk of the work done in the subject is by the use of ready made text-book problems.

The writer has been developing a method and means for presenting certain topics in such a way that all of the pupils are obliged to do some original work, and the present article, written at the request of the editors of The Mathematics Teacher, is illustrated by material which has been found to have value in the class-room both with beginners and with advanced students.

Process examples alone will be considered. It will be recognized that applied problems may easily be treated in a similar manner, and are, in fact, dealt with by means of problems of local interest, constructed by pupils who contribute facts from their own experience, or which relate to their neighborhood or to industries of their town. The value of such a treatment of applied problems is well known and widely recognized. It is commonly used by teachers with great success and much has been published in that line, but there remains a field in which very little has been accomplished but in which there is opportunity for much to be done. When once the value is recognized it is bound to be widely used for teaching mathematics and for creating a new interest in the operations of the science. If imperfectly understood, mathematical operations always create in the mind of the pupil a growing dislike for the subject which prevents sincere effort toward the mastery of the difficulties.

Process examples present to the pupil difficulties both of values and of operations.

* Copyrighted illustrative material by permission of Little, Brown & Co., Boston, Mass., publishers of Examples in Algebra, 1914.

By removing from the statement of a process example all reference to values, a new type of exercise in mathematics is created in which operations alone are indicated. All elements of confusion due to the presence of particular values are avoided.

We will call such a combination of symbols of operation a blank form.

A blank form to suggest the difference of two squares may be written as follows:

$$(\ )^2-(\ )^2.$$

When such a blank form is first presented to a class it does not take long for some one to suggest an answer in blank form such as:

$$[(\ )+(\ )]\,[(\ )-(\ )].$$

An identity in blank form may then be constructed which shows at a glance the essential structure of the difference of two squares expressed as the product of the sum of two numbers multiplied by the difference

$$(\ )^2-(\ )^2\equiv[(\ )+(\ )]\,[(\ )-(\ )].$$

A discussing with the class of the operations involved soon shows the teacher whether or not they know what to do and how to go about the work when special values are substituted in the blanks.

The next step is to require the class to fill in the original blank form in a great variety of ways,

$$(\ )^2-(\ )^2,$$

and to have examples arising in this way solved by other members of the class.

In this way such process examples as the following will be immediately volunteered by the class:

$$x^2-4;\ y^2-9;\ z^2-25;\ 16-a^2;\ 36b^2-c^2.$$

When someone volunteers x^2-3 the difficulty can be immediately attended to and a special prescription adapted to the sufferer in question can be quickly compounded on the spot.

It is natural for the pupils to like to contribute something of

their own and considerable competition develops among the members of the class with everybody trying to invent something new and all hands doing something.

In this way a great variety of examples will be constructed in a few moments and weaknesses on the parts of individuals can be quickly discovered and overcome. The grasp on the whole subject will be strengthened and everyone will be ready and eager for something harder in the line of ready made book problems. They will be capable of doing much more difficult work because they have analyzed the operations and the structure of the examples which they are studying.

It will be found to be interesting to discuss with the class the reason why one of the two blank forms

$$(\quad)^4-(\quad)^2 \text{ and } (\quad)^6-(\quad)^3$$

has rational factors while the other does not, and to suggest changes which may be made so that both expressions or forms will appear as the difference of two squares.

Blank forms help in studying the reduction of fractions to lowest terms. Such blank forms as the following will be found to be valuable:

$$\frac{(\quad)}{(\quad)+(\quad)},$$

$$\frac{(\quad)+(\quad)}{(\quad)^2-(\quad)^2}.$$

After the class has had considerable experience in simplifying fractional forms, the following blank form will be found to fix matters definitely in the mind:

$$\frac{(\quad+\quad)^2-(\quad)^2}{(\quad)^2-(\quad-\quad)^2}.$$

Sums and differences of like powers with all of the confusion incident thereto may be presented in a very simple manner by using blank forms such as

$$x^3+(\quad);\ y^4-(\quad);\ z^5+(\quad).$$

The method applies readily and is especially adapted to irrational forms such as the following:

$$\frac{(\ )}{\sqrt{(\ )}};\qquad \frac{(\ )}{\sqrt{(\ )}};\qquad \frac{\sqrt[3]{(\ )}}{\sqrt{(\ )}};$$

$$\frac{(\ )}{\sqrt{(\ )}+(\ )};\qquad \frac{(\ )}{\sqrt{(\ )}+(\ )}.$$

The confusion which naturally exists in the minds of many pupils between the two last forms may be readily overcome by considering the operations without the distraction which commonly accompanies the presence of particular values.

Many other blank forms relating to rationalization may be invented such as:

$$\frac{\sqrt{(\ )}}{\sqrt{(\ )}+\sqrt{(\ )}},\qquad \frac{\sqrt{(\ )}+\sqrt{(\ )}}{\sqrt{(\ )}-\sqrt{(\ )}}.$$

Partly filled-in blank forms are of great assistance in teaching such topics as trinomial squares.

For example,

$$x^2+6x+(\ )$$

shows whether or not the pupil knows and understands the structure of the trinomial square. By leaving blank one or more of the terms of a trinomial square, it is possible to cause the pupil to examine expressions critically for the necessary and sufficient conditions to be satisfied in order that an expression shall be the square of a binomial.

Thus, such a blank form as

$$(\ )^2-24x+9$$

is a good test of the ability of the pupil to recognize the essentials of the form considered as being the square of a binomial.

Of course it will be recognized that many indeterminate blank forms may be presented, such as,

$$(\ )^2+48xy+(\ )^2.$$

Such a form as this arouses considerable competition among the members of the class in giving various combinations of numbers which may be substituted, and in discussing the results obtained.

Special trinomial forms are readily studied by means of partly filled-in blank forms.

The blank form

$$(\quad)^2 + 5(\quad) + 6$$

will be found to be a good introduction for

$$(a+b)^2 + 5(a+b) + 6,$$

and for others such as

$$(x-y)^2 + 5(x-y) + 6.$$

Indeterminate partly filled-in blank forms such as

$$m^2 + (\quad)m + 20$$

used in factoring will serve to fix the attention of the class on the last term, 20, and will require them to separate 20 into different pairs of factors and to use the proper sums for the coefficient of the second term.

Thus, the attention of the class is necessarily directed to the term which should be first examined in the process of factoring by inspection.

Other blank forms may be used, such for example as

$$a^2 + 6a + (\quad)$$

and

$$b^2 - 7b - (\quad).$$

It will be found to be interesting to discuss with the class the reasons for the possibility of finding a greater number of substitutions which can be used with one of these forms than with the other.

When studying polynomial expressions of the third degree of the form

$$x^3 + ax^2 + bx + c$$

the value of the blank forms will be seen by such preparatory material as

$$a^3 + 14a^2 + (\quad)a + 40,$$
$$b^3 + 13b^2 + (\quad)b + 36,$$
$$c^3 + (\quad)c^2 + (\quad)c - 48,$$
$$d^3 - (\quad)d^2 - (\quad)d - 30.$$

Someone will discover that the set of three factors 8, 5 and 1

and also 2, 2 and 10 of 40 have the same sum, 14, while the sums of the products taken two at a time are different, so that there is an ambiguity and the first form as given is indeterminate.

Then someone will discover the two different substitutions which can be made in the second illustration.

Such examples will show the class the necessity of examining an expression carefully before deciding on a result, and consequently they will see the danger of jumping at conclusions before they have critically examined an expression given them.

Mathematics must train judgment to be of value and mere machine like accuracy is dangerous if not accompanied by careful planning before carrying out some particular operation.

Blank forms are designed to help plan the work. They call attention to the structure and mathematical form.

They are designed to check the natural tendency to start something and then trust to luck that the answer will come somehow.

Many other blank forms may be used to help in the study of polynomial expressions and it will be found without exception that they serve to fix attention on the essentials of the algebraic forms of the topics studied. No class can work with such material without gaining a grasp on the subject which is impossible to some, if not to all, without some means.

Partly filled-in blank forms are of great value in such topics as fractions.

Thus, the fraction

$$\frac{c^2 + 7c + 12}{c^2 + (\ )c + (\ )}$$

when given to the class admits of a variety of substitutions.

The different results obtained should be discussed with the class until many substitutions have been given which admit of simplifying the resulting fraction.

It will be found to be difficult to find a better means at the command of the teacher for insuring the grasp of the principle studied.

Members of a class who possess a little ingenuity will produce a blank form as an answer for the result obtained by performing the operations

$$[\sqrt{(\ )} + \sqrt{(\ )}]^2.$$

The result may appear as

$$(\ ) + 2\sqrt{(\ )(\ )},$$

and someone may suggest the following and give a good reason for it:

$$(\ ) + 2(\ )\sqrt{(\ )(\ )}.$$

A great variety of definite numerical or literal results may be obtained by substituting different values in the above blank form. Thus we may expect from the class such results as:

$$(\sqrt{5} + \sqrt{2})^2,$$
$$(\sqrt{x} + \sqrt{y})^2,$$
$$(\sqrt{2a} + \sqrt{3a})^2.$$

The drill obtained in working these examples will afford a great amount of practice in the class-room and it will become very evident as to just who means business and who does not.

When someone in the class suggests

$$(\sqrt{3} + \sqrt{-3})^2$$

an interesting situation develops which gives the teacher an opportunity to enter the very interesting field of imaginary numbers.

By using blank forms it is possible to present a definite principle or problem to a class, and by holding back one or more of the conditions or elements in the statement of the question, by using blank forms to be filled in, it is possible to hold in check those members of the class whose reaction time in mathematics is short, and who always respond quickly, while others, whose reaction time is longer, become confused and discouraged, because by the time they have grasped the meaning of the question its value has been destroyed for them through its solution by someone else.

Blank forms enable the teacher to ask, "What are you going to do when you know all of the facts?"

Furthermore, they serve to make the class keen to get all of the necessary facts before beginning to operate.

When studying irrational equations blank forms will be found to be very interesting.

Such a group of suggested irrational equations as the following will help in the matter of extra roots:

$$\begin{aligned}
\sqrt{(\ )} + \sqrt{(\ )} + \sqrt{(\ )} &= 0,\\
\sqrt{(\ )} - \sqrt{(\ )} + \sqrt{(\ )} &= 0,\\
\sqrt{(\ )} + \sqrt{(\ )} - \sqrt{(\ )} &= 0,\\
\sqrt{(\ )} - \sqrt{(\ )} - \sqrt{(\ )} &= 0.
\end{aligned}$$

Such a set of equations when filled in with the same numbers and letters in the same columns will help in showing that one of the two values obtained by solving the resulting equations sometimes satisfies one of the given equations; another value often satisfies some other equation; perhaps both values satisfy a third and neither satisfies the remaining equation.

An identity, partly in blank form, will often be of value in studying polynomials as an introduction to certain relations in the theory of equations.

For example,

$$(x + (\ ))(x + (\ ))(x + (\ )) \equiv x^3 + (\ )x^2 + (\ )x + (\ )$$

will help, both in factoring and in the theory of equations.

The relations existing between the roots and coefficients of an equation are readily brought out by a set of equations which are partly in blank form, such as,

$$\begin{aligned}
a^2 + (\ )a + (\ ) &= 0,\\
b^2 - (\ )b + (\ ) &= 0,\\
c^2 + (\ )c - (\ ) &= 0.
\end{aligned}$$

The class may be asked to find, if possible, some other blank form and to determine the signs for another equation suggested by

$$d^2 \quad (\ )d \quad (\ ) = 0.$$

When used to their best advantage blank forms enable the teacher to show what can be done and why it should be done. They train the pupils to try to discover what should be done first. They tend to encourage the careful examination of a problem proposed and the pupils using them will be better qualified to

discriminate between different ways for securing a result and to choose the better way. They encourage originality, and above all, they show the teacher the way the mind of the pupil is acting and they reveal his purpose as to what he proposed to do, and an opportunity is often gained to head off the wrong method and thus to prevent the student from learning the lesson in the wrong way.

HIGH SCHOOL OF COMMERCE,
WORCESTER, MASS.

APPLICATION OF THE PELL EQUATION TO THE PROBLEM OF EXTRACTION OF THE CUBE ROOT OF A BINOMIAL SURD.

By E. E. Whitford.

In the solution of cubic equations and in certain other cases it is desirable to have a somewhat general method for extracting the cube root of a binomial surd. This may be accomplished by the aid of the Pell Equation. To illustrate by an example let it be required to find the cube root of

$$553090\sqrt{2}+53848\sqrt{211}.$$

Let

$$(553090\sqrt{2}+53848\sqrt{211})^{\frac{1}{3}}=m\sqrt{2}+n\sqrt{211}.$$

Then

$$(553090\sqrt{2}-53848\sqrt{211})^{\frac{1}{3}}=m\sqrt{2}-n\sqrt{211}.$$

Multiplying these two equations,

$$2m^2-211n^2=(611817096200-611817098944)^{\frac{1}{3}},$$
$$2m^2-211n^2=(-2744)^{\frac{1}{3}},$$
$$2m^2-211n^2=-14.$$

Let

$$n=2y.$$

Then

$$m^2-422y^2=-7$$

The solution of this generalized Pell equation may be obtained from one of the convergents of $\sqrt{422}$ when developed into a continued fraction,

$$\sqrt{422}=20+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{5}\quad\cdots}}$$

or written in another form*

* For tables of continued fractions and for various methods of solving Pell equations see "The Pell Equation," E. E. Whitford, College of the City of New York, 1912.

$$\sqrt{422} = 20,\ 1,\ 1,\ 5,\ 2,\ 1,\ 3,\ (20)\ \cdots$$
$$1, 22, 19, 7, 14, 23, 11, (2)\ \cdots,$$

where the numbers in the second line denote the values of the form $m^2 - 422y^2$; and hence tell which convergent to select, in this case the fourth. The values of the form are alternately + and —. The convergents are found to be

$$\frac{1}{0},\ \frac{20}{1},\ \frac{21}{1},\ \frac{41}{1}.$$

We select the fourth since for this the value of the form is -7,

$$m = 41.\quad y = 2,$$
$$n = 4.$$

then

$$\therefore (553090\sqrt{2} + 53848\sqrt{211})^{\frac{1}{3}} = 41\sqrt{2} + 4\sqrt{211}.$$

In solving $m^2 - Cy^2 = H$, if $H > 2\sqrt{C}$ certain modifications of the method have to be introduced, but where the solution of the cube root of a binomial surd exists in the form of $m\sqrt{p} \pm n\sqrt{q}$, m, n, p, q, positive integers, this method is always theoretically possible.

Second Illustration:

Extract the cube root of

$$25762\sqrt{2} + 14260\sqrt{7}.$$

(1) Let

$$(25762\sqrt{2} + 14260\sqrt{7}) = m\sqrt{2} + n\sqrt{7};$$

(2) then

$$(25762\sqrt{2} - 14260\sqrt{7})^{\frac{1}{3}} = m\sqrt{2} - n\sqrt{7}.$$

Multiplying (1) and (2)

$$(1327361288 - 1423433200)^{\frac{1}{3}} = 2m^2 - 7n^2;$$
$$\therefore 2m^2 - 7n^2 = (-96071912)^{\frac{1}{3}},$$
$$2m^2 - 7n^2 = -458.$$

Let $n = 2q$.

$$2m^2 - 28q^2 = -458,$$

(3)
$$m^2 - 14q^2 = -229.$$

Now in seeking to solve the equation

(A)
$$m^2 - Cq^2 = H, \text{ if } H > 2\sqrt{C},$$

the roots, m, q, can be found from the roots of a similar equation

$$z^2 - Cy^2 = H_1,$$

by the formulas

$$(B) \qquad m = \frac{K_1 z + Cy}{H_1}, \quad q = \frac{K_1 y + z}{H_1},$$

provided $K_1 < \frac{1}{2}H$, and such that

$$\frac{K_1^2 - C}{H} = H_1, \text{ an integer.}$$

For substituting the values in (B) into equation (A)

$$\frac{K_1^2 z^2 + 2K_1 Czy + C^2 y^2}{H_1^2} - \frac{CK_1^2 y^2 + 2CK_1 yz + Cz^2}{H_1^2} = H,$$

$$\left(\frac{K_1^2 - C}{H}\right) z^2 - C\left(\frac{K_1^2 - C}{H}\right) y^2 = H_1^2,$$

$$z^2 - Cy^2 = H_1.$$

The roots, z, y, can in their turn be found from the roots of a similar equation

$$l^2 - Ck^2 = H_2,$$

and the process can be repeated until the H_i is less than $2\sqrt{C}$.

Pursuing this method in solving equation (3) we seek

$$K_1 < \tfrac{1}{2} \cdot 229$$

and such that

$$\frac{K_1^2 - 14}{-229} = H_1$$

and find

$$\frac{2304 - 14}{-229} = -10;$$

$$\therefore K_1 = 48, \quad H_1 = -10.$$

Now applying this method to

$$z^2 - 14y^2 = -10,$$

seeking $K_2 < \frac{1}{2}\cdot 10$, and such that

$$\frac{K_2^2 - 14}{-10} = H_2,$$

we find

$$\frac{4 - 14}{-10} = 1;$$

$$\therefore K_2 = 2, \quad H_2 = 1.$$

But a solution of

$$l^2 - 14k^2 = 1,$$

is evidently $l = 1, \quad k = 0.$

Then by the formulas

$$z = \frac{K_2 l + Ck}{H_2}, \quad y = \frac{K_2 k + l}{H_2};$$

$$z = \frac{2\cdot 1 + 14\cdot 0}{1} = 2, \quad y = \frac{2\cdot 0 + 1}{1} = 1,$$

and by the formulas (B)

$$m = \frac{48\cdot 2 + 14\cdot 1}{-10}, \quad q = \frac{48\cdot 1 + 2}{-10},$$

$$m = 11, \quad q = 5,$$

disregarding quality sign, and since $n = 2q$, $n = 10$,

$$\therefore (25762\sqrt{2} + 14260\sqrt{7})^{\frac{1}{2}} = 11\sqrt{2} + 10\sqrt{7}.$$

COLLEGE OF CITY OF NEW YORK,
NEW YORK CITY.

NEW BOOKS.

Who's Who in America. Vol. VIII. 1914–1915. Edited by ALBERT NELSON MARQUIS. Chicago: A. N. Marquis & Co. Pp. 2920. $5.00.

This new volume contains 21,459 sketches, of which 4,426 have appeared in no previous edition. Of the sketches in Vol. VII. 933 are not found in Vol. VIII. on account of death and 826 for various other reasons. To collect suitable and accurate data for such a large number of people is a heavy task and the editor deserves credit for the manner in which he has carried this into effect. The importance of the work from an educational standpoint is being more and more recognized and it is finding its way in an ever-increasing number of schools as a reliable source of information concerning our most conspicuous living Americans.

Eighteen Thousand Words Often Mispronounced. By WILLIAM HENRY P. PHYFE. New York: G. P. Putnam's Sons. Pp. 780. $1.50.

This compact little volume is a complete handbook of difficulties in English pronunciation, including a large number of proper names and words and phrases from foreign languages. When there are two accepted pronunciations the authority is given for each. Most persons will find it a very valuable book to have handy on their desk and it should be in every school.

Plane Trigonometry and Tables. By GEORGE WENTWORTH and DAVID EUGENE SMITH. Boston: Ginn & Company. Pp. 292. $1.10.

In preparing this work the authors have carefully considered the demand for thoroughly teachable textbooks in trigonometry. The uses of the subject are clearly set forth in the opening pages, and a large number of practical problems are given as soon as the functions are defined.

The theoretical part of the subject is postponed, whenever possible, until the practical uses are shown and the need for it appears. For example, the graphs of functions, radian measure, and trigonometric equations are treated late in the course where their significance is evident. Among the lines of application that are emphasized are simple surveying and plane sailing, no attempt being made to enter into the technicalities of these subjects.

The topic of logarithms is treated in a special chapter. The tables have been prepared both to four and five decimal places and are arranged on the common system, but conversion tables are introduced allowing for the decimal divisions of the degree.

The greatest care has been taken in the illustrations, the arrangement of topics, the grading of the problems, and the make-up of the book.

Algebraic Invariants. By L. E. DICKSON. New York: John Wiley and Sons. Pp. 97. $1.25 net.

This is No. 14 of the Mathematical Monographs put out by the publishers. It is divided into three parts of about equal length. Part I. treats of linear transformations from the standpoints of both analytic and projective geometry. Invariants and covariants are illustrated and defined as are also Hessians and Jacobians. The uses of the Hessian in the solution of a cubic equation and in the discussion of the points of inflection of a plane cubic curve are given. Part II. treats of the algebraic properties of invariants and covariants in non-symbolic notation. It also treats of homogeneity, weight, annihilators, seminvariant leaders of covariants, law of reciprocity, fundamental systems, properties as functions of the roots, and production as differential operators. Part III. gives an introduction to the symbolic notation of Aronhold and Clebsch. Hilbert's theorem on the expression of the forms of a set linearly in terms of a finite number of forms of the set is proved and applied to establish the finiteness of a fundamental set of covariants of a system of binary forms. There is also a discussion of transvectants and of the types of symbolic factors in any term of a concomitant of a system of forms in three or four variables.

Plane Trigonometry with Tables. By CLAUDE IRWIN PALMER and CHARLES WILBUR LEIGH. New York: McGraw-Hill Book Company. Pp. 288. $1.50 net.

It is refreshing to find now and then a text in trigonometry that defines the trigonometric functions for any angle and then specializes for the acute angle. Why anyone preparing a text for use in colleges or technical schools should want to cover the same ground practically three times instead of once is somewhat of a mystery. The ideas involved in the definitions for general angles would seem to be little if any more difficult than for acute angles and the student instead of being wearied by repetition is interested in seeing how the special cases come out of the general.

Inverse trigonometric functions and trigonometric equations are introduced early and used throughout. The problems are numerous and seem to be carefully selected. There is a chapter on complex numbers, series and hyperbolic functions. The theory and use of logarithmetic and trigonometric tables are taken up rather fully and placed in connection with the tables. The book has many advantages that will appeal to the progressive teacher.

Fergusson's Percentage Trigonometry or Plane Trigonometry Reduced to Simple Arithmetic with a Short Description of His Percentage Compass. By JOHN COLEMAN FERGUSSON. New York: Longmans. Green and Company. Pp. 155. $1.25 net.

This is a remarkable book to those who have never read it or the author's larger work. By means of a percentage unit and two rules

anyone can find the natural sine, cosine, etc., of any angle required, so that the sides and angles of a triangle may be solved by arithmetic. The percentage unit is formed as follows: Describe about a circle a square and join the points of contact, which we may take as vertical and horizontal diameters. Divide each side of the square from the point of contact to the extremities into 100 equal parts and number them in opposite directions from the middle to the ends. Then if the center of the circle is joined to each of these divisions the corresponding octants of the circumference will be divided into 100 unequal parts. Any line joining the center of the circle to any one of the octant divisions will form, together with its adjacent quadrantal line, a percentage angle and the percentage number of the angle itself will always show the ratio of the departure of that particular inclined line to the quadrantal line, from which it is numbered. The length of the quadrantal line or radius, multiplied by the central angle number, which is a decimal, will give the length of the perpendicular or tangent.

Mathematics for Agricultural Students. By HENRY C. WOLFF. New York: McGraw-Hill Book Company. Pp. 309. $1.50 net.

The aims of the author in writing this book are (1) to train the student to do neat and careful work, (2) to encourage further use of elementary algebra and geometry, (3) to develop the habit of careful and logical thinking, (4) to discover good methods, (5) to show how mathematics is helpful in pursuing other subjects of study. The chapter headings are: graphic representation; logarithms; circular functions; ellipse; slide rule; statics; permutations, combinations and binomial theorem; progressions; probability; small errors; point, plane and line in space; maxima and minima; empirical equations. The student who works carefully through the book will know much more mathematics in a useful form than most farmers.

Elementary Mathematical Analysis. By CHARLES S. SLICHTER. New York: McGraw-Hill Book Company. Pp. 490.

This book is intended for first-year college students and covers algebra, trigonometry and conic sections. The idea of functionality is prominent throughout and the work is developed in accordance with the twofold plan of, first, grouping the material around the three functions, $y = ax^n$, $y = a \sin mx$, $y = e^x$, and second, of enlarging the elementary functions by fundamental transformations. This in common with other volumes of the Modern Mathematical Texts published by this company shows a freshness in harmony with the trend for improved mathematical teaching.

Constructive Text-Book of Practical Mathematics. By H. W. MARSH. Vol. III., Technical Geometry. New York: John Wiley and Sons. Pp. 244. $1.25 net.

In the belief that direct contact with demonstrable truth is necessary for the development of reason the author of this book attempts to effect

this contact in the case of geometry by leading the student through constructive and practical work to demonstrative geometry. It would be interesting to see how students taught by a good teacher using this book compared in their knowledge of geometry with students taught by an equally good teacher using the syllabus method. The worth of the book might be determined by such a test.

It could hardly be called good pedagogy to put practically all the definitions, axioms, etc., together at the beginning, or to put as the first proposition to prove that a "straight angle equals 180°."

Examples in Algebra. By A. H. WHEELER. Boston: Little, Brown & Company. Pp. 257. 90 cents.

This splendid collection of problems is adapted for use with any text and is suitable for students preparing for college or technical school. The book has many excellent features, such as its simplicity of arrangement, one-step-at-a-time problems, and the large number of mental exercises. Teachers of algebra will be glad for this new set of 10,000 problems from which they may draw for exercises.

Second Course in Algebra. By W. B. FITE. Boston: D. C. Heath and Company. Pp. 247. 90 cents.

This book is intended for those who have had a year of algebra and gives a careful review of those subjects studied in the first year and then takes up those subjects that properly belong to the second year of study. Simple though natural problems relating to geometry and physics have been supplied and will stimulate the student's interest. The subject of ratio and proportion is treated in connection with fractions which it naturally follows.

Geometry of Four Dimensions. By HENRY PARKER MANNING. New York: The Macmillan Company. Pp. 348. $2.00.

A knowledge of higher mathematics is not necessary to read this book, though the treatment is mathematical. The author has endeavored to build up a structure that will rest on the foundation laid in the schools. The treatment is synthetic, no analytic proofs being given, in the belief that the study of the figures themselves will serve best to give an understanding of them. Analogies have been cherished and used where convenient. The study of four-dimensional geometry will give us a truer view of the nature of geometrical reasoning and increase our power of constructive imaging as well as give us a better conception of the geometries of lower dimensions. Professor Manning has done a good service in giving us this book.

Family Expense Account. By T. A. BROOKMAN. Boston: D. C. Heath and Company. Pp. 112. 60 cents.

Through the somewhat novel and ingenious plan of tracing the financial history of a newly married couple for a series of years, pupils are

taught the value of money, how to keep household accounts, the necessity of planning in order to make their income meet the necessities and at the same time leave any sum of permanent saving.

Arithmetic, domestic economy, household accounting, the writing of checks and other business papers, the placing of insurance, etc., are made very real. While learning these matters young people are also taught certain of the fundamental facts of economic and social life. The book is to be commended because of the service it will render society.

Our Little Spartan Cousin of Long Ago. By JULIA DARROW COWLES. Boston: The Page Co. Pp. 145. 60 cents.

This story of Spartan life and character is a composite picture representative of this unique people and closes during the times of the Persian invasion. The author brings out the true nobility and rugged simplicity of the Spartan character and the book will be found not only interesting but historical. It is a good addition to the Little Cousin Series.

Trigonometry. By MAXIME BÔCHER and H. D. GAYLORD. New York: Henry Holt and Ca. Pp. $142 + ix$.

This book devotes seventy-two small pages to plane trigonometry, and twenty-six pages to spherical trigonometry, the rest being given to sets of exercises which are in a separate division at the back. The tables are not included. It has several interesting features, among which are a considerable use of projections, including the proof of the formulas for $\sin(x+y)$ and $\cos(x+y)$ for the general case, the reduction of the less important parts of the subject to small type so that they may be omitted at the discretion of the teacher, and the enclosing of the answers to numerical problems in small rectangles to distinguish them from intermediate calculation.

Like many other books, its first chapter leaves the pupil with the impression that trigonometric functions exist only for acute angles. It seems as if space could be spared at the beginning to at least mention the fact that the general case exists. The consideration of angles in the form $n(90°) \pm A$ is not very well handled, as it gives the pupil only a method of work for individual cases, instead of expressing the easily formulated general case.

The book may well appeal to many teachers. It is brief, but does not omit essentials, and it is attractive in its make-up.

The Rose of Roses. By MRS. HENRY BACKUS. Boston: The Page Company. Pp. 356. $1.25 net.

"Toni," a German girl of great beauty and the possessor of a wonderful voice, and a young American architect meet in a Kaffee-haus in Bremen where Toni sings every evening. She is anxious to go to America where she hopes to enter a brilliant career as a singer and when the architect goes to bid her good bye he offers her the chance to go by

consenting to "a trial engagement." She accepts the offer and on the way some interesting happenings lead to the usual result.

Plantation Stories of Old Louisiana. By ANDREW WILKINSON. Boston: The Page Company. Pp. 338. $2.00 net.

A charming book of stories after the order of Uncle Remus, told in the quaint negro dialect of the South of years ago. It is a book that will not only please children but those of larger growth.

The Little Florentine. By H. DE CHARLIEU. Boston: The Page Company. Pp. 129. $1.25 net.

A pleasing story for children of a boy who aspired to be a great violinist, but whose father wanted him to become a miller like himself. On going to mill one day the donkey, which the boy was driving, threw off the bags of grain, and as he was not able to replace them he sat down and played his violin. It chanced that the Duke of Guise passed by and was so pleased with his playing that he took him to Paris and after many misfortunes he became one of the King's musicians.

NOTES AND NEWS.

IDEAS on Christmas giving are rapidly changing among the sensible. Those who think as they give are looking for year-round service as the important thing.

In a week of shopping, with all its strain, you will not find a better gift than a year's subscription to the *Youth's Companion*. It offers its service, its clean entertainment, its fine suggestiveness week after week; and the end of the year, which finds many a gift in the attic, dust-covered and forgotten, brings *The Companion* again, with all the charm of last Christmastide.

No American monthly at any price offers the same amount of reading, and none can offer better quality. Less than four cents a week provides this best of Christmas gifts—$2.00 a year. If you subscribe now, all the remaining issues of the year will be sent free, and The Companion Home Calendar. A copy of the Calendar is also sent to those who make a gift subscription. Send for sample copies, and the Forecast for 1915. The Youth's Companion, 144 Berkeley Street, Boston, Mass.

The Century Magazine announces that, beginning with the November number, it will interpret to its readers what lies back of the bare facts of bulletin and dispatch.

In New York, W. Morgan Shuster, author of "The Strangling of Persia," and Samuel P. Orth, professor of politics and public law at Cornell University, will write of the war and its effects from an ethnic and political point of view.

In London, James Davenport Whelpley, author of "The Trade of the World," and of recent *Century* papers on diplomacy, will deal with the personalities of the war lords and the spirit of the nations.

Estelle Loomis, the short story writer, now in Paris, will be sending *The Century* sketches of human interest.

Albert Bigelow Paine, author of "Mark Twain: A Biography," now in Switzerland, has put himself in touch with the

German authorities with a view to presenting views of Germany in war time.

Dr. Hendrik Van Loon, historian and journalist, has gone to his native land, Holland, where he will write of the Lowlands in war time.

Albert Edwards, the well-known novelist and travel writer, is going to Europe with a roving commission from *The Century Magazine*. His mission is to help build the literature that will grow out of the war itself.

EASILY the most important and interesting of the features planned for the new year of *St. Nicholas* is the new serial by Frances Hodgson Burnett, "The Lost Prince," which will begin in the November number and run through the entire year.

Mrs. Burnett has based her new story on a legend in a certain European country that, five hundred years ago, while a ruthless and unpopular king was in power, his son, a youth of noble qualities and much beloved by his people, mysteriously disappeared. And, upon the death of his father, a new dynasty came to the throne though the story of the lost prince has been handed down through the centuries.

PROBABLY the most distinctive feature of the New International is the amount of encyclopedic information that it contains. Wherever the reader turns he finds admirably condensed treatises, or tables, or illustrations. It is impossible to use the New International without being continually surprised by the range and completeness of the information furnished. The type matter is equivalent to that of a 15-volume encyclopedia.

For example, most of us know the bare fact that malaria is spread by mosquitoes. If we look up *Malaria,* or *mosquito,* we find, following an excellent definition, a cross reference to *Anopheles,* the genus of malarial mosquitoes. Under *Anopheles* is not only a discussion of the spreading of malaria by mosquitoes, but a description and an illustration by which you are enabled to tell at a glance whether the particular mosquito resting upon your sleeve is of the malarial variety.

To more important terms much more space is devoted (as *star,* two columns; *man,* two pages; *automobile,* three pages)

so that their definitions are perfect mines of information. So completely does the New International cover all realms of knowledge that its possessor requires no other work of general reference. It has well been called, next to the Bible, *the most valuable book in the English Language.*

THE fall meeting of the Philadelphia Section was held in the Music Room of the William Penn High School for Girls Wednesday, October 28, 1914, at 3 P. M. with the following program: Paper: The Mechanics of Aviation, L. P. Eisenhart, Ph.D., Princeton University; Paper: The Relation of Elementary Mathematics to that of the Higher Schools, J. Wilbur Haines.

THE October number of the *Monist* contains an article on *The Principles of Mechanics with Newton From 1679 to 1687,* an article on *Pandiagonal Prime Number Magic Squares,* and one on *Paneled Magic Squares.*

NEW MEMBERS.

LOCK, L. LELAND, 950 St. Johns Place, Brooklyn, N. Y.
MASTERS, FREDERICK G., 3444 Bouquet St., Pittsburgh, Pa.
MILLER, ABRAM B., 608 13th St., Altoona, Pa.
RAYNSFORD, MISS HELEN J., 32 Upton Park, Rochester, N. Y.
SPARKS, S. WALTER, c/o H. F. Hart, 67 Valley Road, Montclair, N. J.
SWICK, WM. A., 1314 Eighth Ave., Beaver Falls, Pa.
WALKER, J. EUGENE, 1420 Pine St., Philadelphia, Pa.
WHITE, PROF. H. S., Vassar College, Poughkeepsie, N. Y.
WOLFENDEN, EMMA, 5901 Ridge Ave., Roxborough, Philadelphia, Pa.
WEBBER, W. PAUL, 316 McKee Place, Pittsburgh, Pa.
QUINN, JOHN J., 2521 Elba St., Pittsburgh, Pa.
INGRAM, EDWARD T., 110 Hoffman Ave., Trenton, N. J.
MCCABE, FLORENCE E., 1000 Packer Ave., Pittston, Pa.
WOLFE, MISS FRANCES R., Danieltown, Va.
BRATTON, KATHARINE M., Elkton, Md.
THAYER, WILLIAM MORTIMER, 821 Belmont Ave., Collingswood, N. J.
BRINSFIELD, MISS ARAMINTA, Rock Hall, Kent Co., Md.
BOWLUS, EDGAR STANLEY, 4217 Swiss Ave., Dallas, Texas.
PARKHILL, MARTHA ELLA, 506 Third Ave., Asbury Park, N. J.
DOUGHERTY, HARRY R., c/o N. Y. Mil. Acad., Cornwall-on-Hudson, N. Y.
MCNUTT, NELLIE IRENE, Maryland College, Lutherville, Md.
BRAGG, W. N., Fort Ann, N. Y.

THE twenty third meeting of the Association was held at the College of the City of New York on Saturday, November 28, 1914. The meeting was notable for the excellence of the papers read and for the interst of those present. The discussion was spontaneous, enthusiastic, and excellent. The writer does not remember a better meeting. The addresses and reports of committees will be available to members through the TEACHER. It is unfortunate that the discussion will not be available.

The following amendment to paragraph 3, article 3, of the constitution was adopted, "There shall be an editorial committee composed of an editor-in-chief, a circulation manager, and as many other members as the Council shall each year deem necessary. The term of each member shall be three years. The members shall be appointed by the Council, in rotation, in successive years."

Latest Mathematical Textbooks

Bôcher and Gaylord's Trigonometry

By MAXIME BÔCHER, Professor in Harvard University, and H. D. GAYLORD, Master in Browne and Nichols School, Cambridge. ix+142 pp. Narrow 12mo. $1.00.

Intended as a textbook for the ordinary course in trigonometry in high schools, technical schools, and colleges. Brevity has been secured without sacrifice of clearness, first by omitting unessential subjects, and secondly by avoiding long-winded explanations of simple matter.

Dowling and Turneaure's Analytic Geometry

By L. W. DOWLING, Associate Professor of Mathematics, and F. E. TURNEAURE, Dean of the College of Engineering in the University of Wisconsin. (*American Mathematical Series.*) xii+266 pp. Large 12mo. $1.60.

Because of its fundamental importance, the concept of functional correspondence and the method of representing such correspondence geometrically has been introduced early in the course. The standard forms of equations of important loci are also developed early, and the properties of these loci discussed later by means of the equations already at hand.

Snyder and Sisam's Analytic Geometry of Space

By VIRGIL SNYDER, Professor in Cornell University, and C. H. SISAM, Assistant Professor in the University of Illinois. (*American Mathematical Series.*) xi+289 pp. 12mo. $2.50.

This book is for beginners, no knowledge of geometry of more than two variables being presupposed.

Henry Holt and Company

34 West 33d Street, NEW YORK **623 South Wabash Avenue, CHICAGO**

The following officers and members of the Council were elected:

President—Eugene R. Smith, The Park School, Baltimore, Md.

Vice-President—Paul Saurel, of the City of New York.

Secretary—F. Eugene Seymour, State Normal School, Trenton, N. J.

Treasurer—Edward D. Fitch, The DeLancey School, Philadelphia, Pa.

Member of the Council—Herbert E. Hawkes, Columbia University, New York City; Harrison E. Webb, Central High School, Newark, N. J.; Miss Catharine D. Lewis, Baltimore, Md.

Dr. J. T. Rorer, Wm. Penn High School, Philadelphia, Pa., was appointed chairman of a committee on revision of the constitution. All suggestions for revision should be sent to him.

TEXT-BOOKS OF TRIGONOMETRY TABLES OF LOGARITHMS

By EDWIN S. CRAWLEY

Professor of Mathematics in the University of Pennsylvania

ELEMENTS OF PLANE AND SPHERICAL TRIGONOMETRY.
Fourth edition, newly revised, vi and 190 pages, 8vo. **Price, $1.10**

THE SAME, WITH FIVE-PLACE TABLES (as below), half leather. **Price, $1.50.**

This book is intended primarily for college use, but is used in many secondary schools also.

SHORT COURSE IN PLANE AND SPHERICAL TRIGONOMETRY.
121 pages, 8vo. **Price, $0.90.**

This book is intended primarily for use in secondary schools, but many colleges, in which the time allotted to trigonometry is restricted, have adopted it.

THE SAME, WITH FIVE-PLACE TABLES. Price, $1.25.

TABLES OF LOGARITHMS, to five places of demicals. Seven tables, with explanations. xxxii and 76 pages, 8vo. **Price, $0.75.**
These tables are arranged in the most approved manner.

ONE THOUSAND EXERCISES IN TRIGONOMETRY. With answers. vi and 70 pages. **Price, $0.50.**
The exercises in this collection are with very few exceptions of an average grade of difficulty, quite suitable for ordinary class room drill and examination. The exercises are classified so that the book can readily be used to supplement any text book. A pamphlet of sample pages will be sent to any teacher desiring it.

Orders, and requests for books for examination with a view to introduction, should be directed to

EDWIN S. CRAWLEY, University of Pennsylvania, Philadelphia

N. B.—In all cases specify by full title which book is desired.